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Resonant sideband instabilities in wave propagation in fluidized beds

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We consider possible mechanisms for the observed secondary instability of travelling instability waves in liquid fluidized beds. The resonance conditions for quadratic nonlinear interaction are solved in the long-wave limit by perturbation theory. The estimated horizontal wavenumber of the resonant sideband agrees well with experiment and supports the proposed mechanism.

1. Introduction

In an earlier paper (Didwania & Homsy 1981), we reported the existence of several flow regimes in liquid fluidized beds. These include (in order of increasing u/u_{mf}) the wavy, turbulent and bubbly regimes. The wavy regime is complex and characterized by a complicated horizontal structure. In this regime, we observed voidage fluctuations in the form of travelling waves. The organization of the wave motion depends upon the flow rate and the distance from the distributor section. For flow velocities close to that of minimum fluidization, the voidage fluctuations have a planar-wave form. As this wavetrain travels upwards, it develops a transverse structure while the vertical wavenumber remains unaffected. With further increase in flow rate, both the horizontal and vertical wavelengths decrease and the voidage fluctuations become increasingly wavy. Figure 1 (taken from Didwania & Homsy 1981) shows this development.

Several investigators (Jackson 1963; Pigford & Baron 1965; Anderson & Jackson 1968; Homsy, El-Kaissy & Didwania 1980) analysed the transition from the state of uniform fluidization to the wavy regime, using a linear stability analysis. The analysis predicts an exponentially growing *planar* wavetrain and describes its experimentally measured growth and propagation properties in the early stages of growth. From their measurements, Homsy *et al.* (1980) deduced values of material constants appearing in the modelling equations.

The mechanism by which the planar train develops the transverse structure or loses its stability is not presently understood. The linear theory predicts the voidage waves to maintain their planar character as they traverse the entire bed height, contrary to our experimental observations. This suggests that the planar wavetrain may itself be unstable, and a secondary instability mechanism leads to the appearance of waves with finite horizontal wavelength, and ultimately to the homogeneous turbulent state.

A plausible mechanism for this type of secondary instability can be examined in the framework of a weak nonlinear theory. Initially planar voidage waves grow exponentially in amplitude as they travel upward. Once their amplitude becomes appreciable, nonlinear effects can no longer be neglected. As a consequence of nonlinearity and weak dispersion, this primary planar mode is capable of resonant

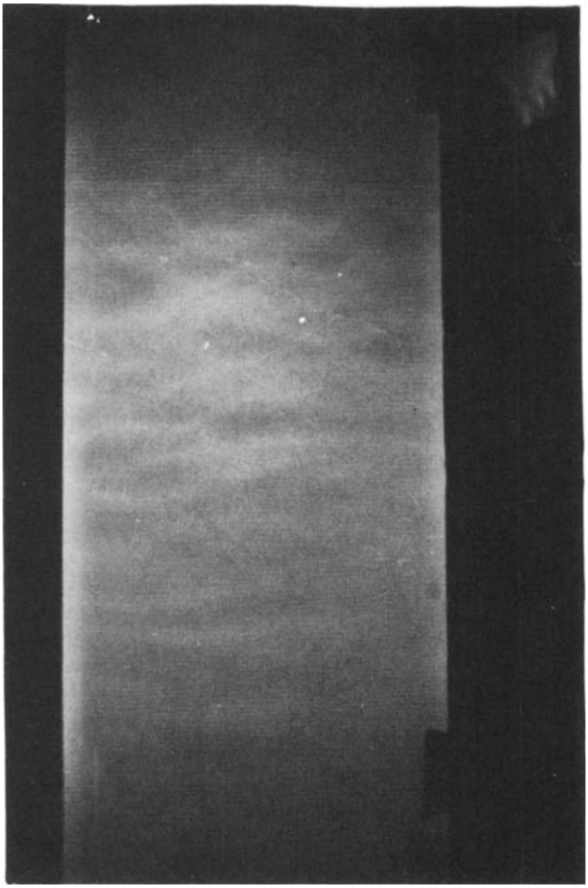

FIGURE 1. A visualization of the wavy regime in a liquid fluidized bed (from Didwania & Homsy 1981).

interaction, *via* its second harmonic, with a two-dimensional disturbance composed of a finite number of modes. Finally, the transverse structure appears as the planar wavetrain loses its stability to this new disturbance packet. This mechanism is analogous to the one responsible for the instability of a periodic progressive wavetrain on deep-water waves. In the later case, it is known that a disturbance capable of gaining energy from the primary wave motion consists of a pair of wave modes at sideband frequencies and wavenumbers fractionally different from those of the fundamental (Benjamin & Feir 1967).

Thus, from the preceding discussion it is clear that, for such a secondary instability mechanism to exist, the observed horizontal wave structure must be capable of resonant interaction with the initially planar waves. Our objective in the present work is to establish the resonance conditions and examine if they are indeed satisfied. These conditions predict the horizontal wavenumber of the sideband modes, and this may be compared with experiment. In §2 we discuss briefly the equations governing the two-dimensional small voidage disturbances to the state of uniform fluidization and obtain the linear dispersion relationship. The resonance conditions and an expression for the horizontal wavenumber of the secondary instability are derived in §3. We compare the computed values of the horizontal wavenumber with our experimentally observed results in §4.

2. Dispersion relation

The two-fluid modelling equations for fluidized beds have been discussed earlier by Homsy *et al.* (1980). Neglecting the fluid phase viscosity terms compared with those of the solid phase, we obtain the following set of dimensionless equations.

Continuity:

$$\left.\begin{aligned} &\text{fluid} \quad \frac{\partial \epsilon}{\partial t}+\frac{\partial(\epsilon u_k)}{\partial x_k}=0; \\ &\text{solid} \quad \frac{\partial(1-\epsilon)}{\partial t}+\frac{\partial(1-\epsilon)v_k}{\partial x_k}=0. \end{aligned}\right\} \tag{2.1}$$

Momentum:

$$\text{fluid} \quad R\left[\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right]=-\frac{\partial p_f}{\partial x_i}-\frac{\hat{\alpha}_3 R}{Re}(u_i-v_i)$$
$$-\hat{\alpha}_4 R\left\{\left(\frac{\partial u_i}{\partial t}+(u_j-v_j)\frac{\partial u_i}{\partial x_j}\right)-\left(\frac{\partial v_i}{\partial t}+(u_j-v_j)\frac{\partial v_i}{\partial x_j}\right)\right\}-R\,Fr\,\delta_{i3}; \tag{2.2}$$

$$\text{solid} \quad (1-\epsilon)\left[\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}\right]=R(1-\epsilon)\left[\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right]$$
$$+\frac{R}{Re}\left\{\frac{\partial}{\partial x_i}\left(\beta^2\frac{\partial u_i}{\partial x_j}+\eta^2\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2}{3}\frac{\partial u_j}{\partial x_j}\right)\right)+\hat{\alpha}_3(u_i-v_i)\right\}$$
$$-\frac{\partial(1-\epsilon)(p_s-p_f)}{\partial x_i}-(1-\epsilon)(1-R)\,Fr\,\delta_{i3}$$
$$+\hat{\alpha}_4 R\left(\left(\frac{\partial u_i}{\partial t}+(u_j-v_j)\frac{\partial u_i}{\partial x_j}\right)-\left(\frac{\partial v_i}{\partial t}+(u_j-v_j)\frac{\partial v_i}{\partial x_j}\right)\right). \tag{2.3}$$

Here u_i, v_i are respectively fluid and solid phase velocities, p_f and p_s are the respective phase pressures and ϵ is the voidage. β^2 and η^2 are respectively the bulk and shear viscosity of the solid phase. The coordinate axes are chosen such that (x_1, x_3) are in the horizontal and vertical directions respectively.

In addition, the dimensionless parameters and material constants are

$$\left.\begin{aligned} &R=\frac{\rho_f}{\rho_s}\ \text{(density ratio)}, \quad Re=\frac{u_0 d_p \rho_f}{\mu_f}\ \text{(particle Reynolds number)}, \\ &Fr=\frac{g d_p}{u_0^2}\ \text{(particle Froude number)}, \quad \hat{\alpha}_3=\alpha_3\frac{d_p^2}{\epsilon\mu_f}\ \text{(drag function)}, \\ &\hat{\alpha}_4=\frac{\alpha_4}{\epsilon\rho_f}\ \text{(virtual mass coefficient)}. \end{aligned}\right\} \tag{2.4}$$

These equations possess a simple steady solution, known as the state of uniform fluidization. It is given dimensionlessly as

$$u_j=\delta_{3j}, \quad v_j=0, \quad \epsilon=\epsilon_0, \tag{2.5}$$

provided that

$$\hat{\alpha}_3(\epsilon_0)=\frac{Re(1-\epsilon_0)(1-R)\,Fr}{R}. \tag{2.6}$$

As in the previous work, we assume an expression for the drag function of the form

$$\hat{\alpha}_3 = \frac{Re(1-\epsilon)(1-R)\,Fr}{R}\frac{u}{u_t}\epsilon^{1-n}, \tag{2.7}$$

where n is the Richardson–Zaki exponent and u_t is the terminal velocity of a single particle. The linearized equation governing the small voidage disturbances ϵ' to this state of uniform fluidization can be readily obtained after some algebraic manipulation:

$$\left(\frac{\hat{\alpha}_4}{\epsilon_0(1-\epsilon_0)}+\frac{1}{R}+\frac{(1-\epsilon_0)}{\epsilon_0}\right)\frac{\partial^2\epsilon'}{\partial t^2}+\frac{\partial \epsilon'}{\partial x_i\,\partial t}\delta_{i3}\left(\frac{\hat{\alpha}_4(2-\epsilon_0)}{\epsilon_0(1-\epsilon_0)}+\frac{2(1-\epsilon_0)}{\epsilon_0}\right)+\frac{(1-R)\,Fr}{R\epsilon_0}\frac{\partial \epsilon'}{\partial t}$$
$$-\frac{\xi^2}{Re\,(1-\epsilon_0)}\frac{\partial^3\epsilon'}{\partial x_i\,\partial x_i\,\partial t}-\frac{M}{R}\frac{\partial^2\epsilon'}{\partial x_i\,\partial x_i}+\frac{(\hat{\alpha}_4+(1-\epsilon_0))}{\epsilon_0}\frac{\partial^2\epsilon'}{\partial x_3\,\partial x_3}$$
$$-\frac{(R-1)\,Fr[(1-\epsilon_0)+(n-1)\,\epsilon_0](1-\epsilon_0)}{\epsilon_0^R}\frac{\partial \epsilon'}{\partial x_3}=0. \tag{2.8}$$

Here $$\xi^2=\beta^2+\tfrac{4}{3}\eta^2, \quad M=-(1-\epsilon_0)\left.\frac{\partial(p_s-p_f)}{\partial \epsilon}\right|_{\epsilon=\epsilon_0}.$$

Solutions exist of the form

$$\epsilon'=\epsilon\exp(\sigma t+i(k_x x+k_z z)). \tag{2.9}$$

Substitution of this form in (2.8) yields the following quadratic algebraic equation for σ in terms of k_x, k_z and other parameters:

$$A\sigma^2+(Bik_z+C(k_x^2+k_z^2)+D)\,\sigma+(Ek_x^2+Fk_z^2+Gik_z)=0, \tag{2.10}$$

where $$A=\frac{\hat{\alpha}_4}{(1-\epsilon_0)\,\epsilon_0}+\frac{1}{R}+\frac{(1-\epsilon_0)}{\epsilon_0}, \quad B=\frac{2(1-\epsilon_0)}{\epsilon_0}+\frac{\hat{\alpha}_4(2-\epsilon_0)}{\epsilon_0(1-\epsilon_0)},$$
$$C=\frac{\xi^2}{Re\,(1-\epsilon_0)}, \quad D=\frac{(1-R)\,Fr}{R\epsilon_0}, \quad E=\frac{M}{R},$$
$$F=\frac{M}{R}-\frac{(\hat{\alpha}_4+(1-\epsilon_0))}{\epsilon_0}, \quad G=\frac{(1-R)\,Fr((n-1)\,\epsilon_0+(1-\epsilon_0))(1-\epsilon_0)}{\epsilon_0 R}.$$

Roots of (2.10) give the dispersion relation. σ is taken to be complex, $\sigma=\sigma_r+i\sigma_i$; σ_r is the growth constant and σ_i the frequency. The wavevector $\mathbf{k}=(k_x, k_z)$ is real; k_x and k_z are respectively horizontal and vertical wavenumbers.

The primary instability is characterized by those wavenumbers giving the maximum growth constant. It is easy to show that $\partial\sigma_r/\partial k_x=0$ at $k_x=0$, and thus the primary instability is a planar wave (Anderson & Jackson 1968). For the primary mode, we also have

$$\left.\frac{\partial \sigma_r}{\partial k_z}\right|_{k_x=0,\,k_z=k_z^*}=0, \tag{2.11}$$

where k_z^* is the experimentally observed vertical wavenumber of the planar wavetrain. The material constants ($\hat{\alpha}_4$, M, ξ^2) appearing in (2.10) were earlier deduced from measured growth and propagation properties of these planar voidage waves. These voidage waves were also found to be dispersive, i.e. the frequency σ_i depended on the wavevector (El-Kaissy & Homsy 1976; Homsy *et al.* 1980).

3. Resonance conditions

As noted earlier, our experimental observations indicate that the vertical wavenumber of the planar wavetrain remains unaffected while a transverse structure characterized by a small horizontal wavenumber appears. These waves with transverse structure can be described as having a wavevector $\mathbf{k}_T = (\pm\gamma, k_z^*)$, where γ is a small quantity. Owing to the growth of the planar waves to appreciable amplitude, the transverse mode $\mathbf{k}_T$ may be able to gain energy from the primary wave $(0, k_z^*)$ *via* its second harmonic $(0, 2k_z^*)$. This assumes that the essential nonlinearity is quadratic. For such a mechanism to be possible, the following resonance condition must hold:

$$\sigma_i(\gamma, k_z^*)+\sigma_i(-\gamma, k_z^*) = \sigma_i(0, 2k_z^*). \tag{3.1}$$

The resonance condition (3.1) has been discussed earlier in the context of weak quadratic nonlinearity by Phillips (1974).

Since γ is small, we obtain by Taylor-series expansion of (3.1) the following expression for the horizontal resonance wavenumber:

$$\gamma^2 = \frac{\sigma_i(0, 2k_z^*)-2\sigma_i(0, k_z^*)}{(\partial^2\sigma_i/\partial k_x^2)|_{k_x=0,\, k_z=k_z^*}}. \tag{3.2}$$

An expression for $(\partial^2\sigma_i/\partial k_x^2)|_{k_x=0,\, k_x=k_z^*}$ can be obtained from (2.10), after some algebraic manipulation. Differentiating (2.10) once with respect to k_x we get

$$2A\sigma\frac{\partial\sigma}{\partial k_x}+2Ck_x\sigma+(\beta ik_z+C(k_x^2+k_z^2)+D)\frac{\partial\sigma}{\partial k_x}+2Ek_x = 0. \tag{3.3}$$

Differentiating (3.3) with respect to k_x we have

$$2A\left(\frac{\partial\sigma}{\partial k_x}\right)^2+2A\sigma\frac{\partial^2\sigma}{\partial k_x^2}+2C\sigma+4Ck_x\frac{\partial\sigma}{\partial k_x}+(\beta ik_z+C(k_x^2+k_z^2)+D)\frac{\partial^2\sigma}{\partial^2 k_x}+2E = 0. \tag{3.4}$$

Separating (3.3) and (3.4) into real and imaginary parts and using (2.11), we obtain as noted earlier,

$$\left.\frac{\partial\sigma_i}{\partial k_x}\right|_{k_x=0,\, k_z=k_z^*} = 0. \tag{3.5a}$$

In addition

$$\left.\frac{\partial^2\sigma_i}{\partial k_x^2}\right|_{k_x=0,\, k_z=k_z^*} = \frac{2[(2A\sigma_i+\beta k_z^*)(C\sigma_r+E)-C\sigma_i(2A\sigma_r+D+Ck_z^*)^2]}{(2\sigma_r A+D+Ck_z^{*2})^2+(2A\sigma_i+Bk_z^*)^2}, \tag{3.5b}$$

where σ_i and σ_r appearing in the above expression are evaluated at $(0, k_z^*)$. Substituting (3.5) in (3.2) yields an expression for γ in terms of $\sigma_i(0, k_z^*)$, $\sigma_i(0, 2k_z^*)$. $\sigma_r(0, k_z^*)$, the material constants and operating parameters.

4. Results and discussion

Equation (3.2) was used to evaluate numerically the values of γ. The material constants entering into the above equation were taken from Homsy *et al.* (1980). These material constants were used to evaluate $\sigma_i(0, 2k_z^*)$ from the dispersion relation and finally γ, the horizontal wavelength.

The numerical values of γ for the experimental conditions of sets A and B, as reported in Didwania & Homsy (1981), always fell in the range 0·01–0·1. The experimentally observed values of γ, obtained through visualizations and optical scanning, are in the range 0·04–0·07. This compares well with the predicted range, considering the uncertainties involved in the estimation of material constants. Such an agreement strongly suggests the existence of the secondary instability mechanism that we have proposed here. A stability analysis of the weakly nonlinear planar wavetrain to the disturbances admitted by the resonance conditions that we have suggested here remains to be investigated.

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